

# A HIGH RESOLUTION SPECTRAL MODEL FOR FLOW IN COMPLEX TERRAIN

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## 1. INTRODUCTION

Jackson and Hunt (1975) and Mason and Sykes (1979) developed semi-analytical theories for the perturbation of the boundary layer flow caused by low hills. These developments have lead to a number of models for neutral flow in complex terrain (Mason and King, 1985; Taylor et. al. 1983; Walmsley et. al. 1982, 1986; Beljaars et. al. 1986).

The models are all based on linearization of the equations of motion and the use of Fourier transforms and independent calculation of velocity perturbations for each wavenumber vector. This decomposition allows for the treatment of flow over real terrain provided the slopes are not so steep as to cause separation. Field experiments (Mason and King (1985), Taylor and Teunissen (1985) and Taylor et. al. (1985)) have demonstrated that these models work very well for the calculation of hilltop profiles of mean wind speed provided that the approach winds are high enough to ensure near neutral conditions. These models are thus well suited for use in connection with wind energy resource assessment in complex terrain.

The linearized equations of motion in cartesian coordinates for neutral flow perturbations relative to a reference state with constant (height independent) horizontal wind velocity  $\mathbf{u}_0$  read

$$\begin{aligned} u_0 \frac{\partial u}{\partial x} + v_0 \frac{\partial u}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial z} \\ u_0 \frac{\partial v}{\partial x} + v_0 \frac{\partial v}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial \tau_y}{\partial z} \\ u_0 \frac{\partial w}{\partial x} + v_0 \frac{\partial w}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\partial \tau_z}{\partial z} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0. \end{aligned} \quad (1)$$

Except for a thin layer near the surface the stress terms can be neglected and the resulting solution for the flow becomes identical to the solution for an inviscid fluid. This is a main result of the Jackson and Hunt analysis. A different analysis is presented by Troen and de Baas (1986). In the latter analysis the stress terms in the Fourier transform of Eq. 1 were approximated by  $K$ -closure with  $K$  dependent on the wavenumber  $\mathbf{k}$ . This approach is consistent with the approaches of the above cited papers, where height independent values for the advective velocities are assigned for each wavenumber.

The analysis showed that for each wavenumber two solutions appear. One solution which is identical to an inviscid solution of a vertical extent  $\propto L \equiv |\mathbf{k}|^{-1}$  independent of the value of the diffusivity  $K$ , and one of much smaller vertical extent  $\ell$  with  $\ell \ln(\ell/z_0) \propto L$ . In this paper a model is described, which builds on these developments.

## 2. THE MODEL

The velocity potential for inviscid flow expressed in cylindrical coordinates can be written as a sum of terms of the form:

$$\chi = \alpha_{na} J_n(ar) \exp(in\phi) \exp(-az), \quad (2)$$

where  $J_n$  is the  $J$ -Bessel function of order  $n$ . From this expression solutions for the horizontal velocity perturbation  $\mathbf{u} = \nabla \chi$  are obtained in the form

$$\begin{aligned} \mathbf{u} = \alpha_{na} \left\{ a \left[ J_{n-1}(ar) - \frac{n}{ar} J_n(ar) \right] \mathbf{e}_r + \frac{in}{r} J_n(ar) \mathbf{e}_\phi \right\} \\ \times \exp(in\phi) \exp(-az), \end{aligned} \quad (3)$$

where  $\mathbf{e}_r$  and  $\mathbf{e}_\phi$  are unit vectors in the radial and azimuthal direction respectively. For the centre point  $r \equiv 0$  this simplifies to

$$\mathbf{u} = \alpha_{1a} \frac{1}{2} a [\mathbf{e}_r + i\mathbf{e}_\phi] \exp(-az), \quad (4)$$

or

$$\mathbf{u} = \alpha_{1a} a \frac{1}{2} (1, i) \exp(-az). \quad (5)$$

A natural boundary condition for distances far away from the area of interest for  $r = R$  say is  $\nabla \chi = \mathbf{0}$ . With this assumption the general solution becomes of the form:

$$\mathbf{u} = \frac{1}{2R} (1, i) \sum_j \alpha_j c_j \exp(-c_j \frac{z}{R}), \quad (6)$$

where  $c_j$  is the  $j$ 'th zero of  $J_1$  and  $\alpha_j \equiv \alpha_{1c_j}$  are arbitrary expansion coefficients. The outer length scale, equal to the decay length in the exponential, becomes  $L_j = R/c_j$ . The coefficients  $\alpha_j$  are determined by the kinematic boundary conditions at the surface  $z = h(r, \phi)$ :

$$w_0 \equiv \left. \frac{\partial \chi}{\partial z} \right|_0 = \mathbf{u}_0 \cdot \nabla h, \quad (7)$$

The radial functions in Eq. 2 form a set of orthogonal functions, and expansions in terms of these functions are usually termed Fourier-Bessel series. The coefficients  $\alpha_j$  are therefore obtainable directly by projection, combining Eqs. 2 and 7. The latter is linear in the upwind reference wind velocity, and in order to efficiently calculate the flow perturbation for all upwind directions two sets of expansion coefficients are calculated:  $\alpha_j^x$  corresponding to  $\mathbf{u}_0 = (1, 0)$  and  $\alpha_j^y$  corresponding to  $\mathbf{u}_0 = (0, 1)$ . Multiplication by  $\exp(-i\phi)$ , integration over azimuth and using the inversion formula given in Oberhettinger (1973) gives:

$$(\alpha_j^x, \alpha_j^y) = -\frac{2R}{c_j J_2^2(c_j)} \int_0^1 \frac{r}{R} J_1\left(c_j \frac{r}{R}\right) \times \left[ \int_0^{2\pi} \nabla h(r, \phi) \exp(-i\phi) d\phi \right] d\frac{r}{R}. \quad (8)$$

The solution for the inviscid flow perturbation for an arbitrary surface and reference wind profile are then given by Eqs. 6 and 8 with  $\alpha_j = u_0 \alpha_j^x + v_0 \alpha_j^y$ . Here the reference wind speed can be set independently for each term in the expansion as in Jackson and Hunt (1975), Mason and Sykes (1979) and Troen and de Baas (1986). In place of the scale height  $L \equiv |\mathbf{k}|^{-1}$  the appropriate scale height here becomes  $L \equiv R/c_j$ .

The inviscid flow solution corresponds to a balance between advection of momentum and pressure gradient force. Near the surface, in a layer of vertical extent  $\propto \ell$ , the stress terms also becomes important. The resultant mean wind profile can be considered as consisting of three parts:

- 1)  $z \gg \ell$  The inviscid solution is appropriate. The relevant scale velocity is  $|\mathbf{u}_0(L)|$ . The perturbation decays exponentially with the decaylength  $L$ .
- 2)  $z \propto \ell$  In the range from  $\ell$  to a few times  $\ell$  the stress terms can be neglected in the dynamics, but the wind shear in the upwind profile becomes important. The velocity scale entering into the pressure perturbation is still  $|\mathbf{u}_0(L)|$ , but the scale velocity in the advection terms is  $|\mathbf{u}_0(z)|$ . The perturbation is modelled as under 1), but using  $(u_0, v_0) = \mathbf{u}_0(L)|\mathbf{u}_0(L)|/|\mathbf{u}_0(z)|$ <sup>1</sup>.
- 3)  $z \leq \ell$  Here the vertical variation of the pressure gradient can be neglected (because  $\ell \ll L$ ). Analytical solutions for the perturbation profile in this layer based on  $K$ -closure was developed by Jackson and Hunt (1975) and incorporated into the numerical models described by Taylor et al (1983). A serious problem with these "inner layer" solutions is that they do not satisfy the condition of asymptotically approach to the inviscid "outer" solutions. In this thin layer the perturbation must approach zero at the surface and grow to match the solution above. The perturbation

speed is observed to vary nearly logarithmic with  $z$ , and the direction cannot deviate much from that of the inviscid solution, again because of the thinness of the layer. The simplest possible way to model this is to use the velocity scale at height  $\ell$  as under 2) and to set the relative perturbation velocity constant for  $z \leq \ell$ . This approach is adopted here.

The model is now fully specified except for  $\ell$ . The analysis of Jackson and Hunt (1975) and Troen and de Baas (1986) gives:

$$\ell \ln(\ell/z_0) = a_1 L, \quad (9)$$

where  $a_1$  is an arbitrary constant of order unity. Eq. 9 is well approximated by the power law:

$$\ell/z_0 = a_2 (L/z_0)^{0.9}. \quad (10)$$

Different arguments, based on the behaviour of the stress profile near the ground, put forward by Jensen (Jensen et al (1984), Taylor et al (1987)) leads to the expression:

$$\ell \ln^2(\ell/z_0) = a_3 L, \quad (11)$$

equivalent to the power law:

$$\ell/z_0 = a_4 (L/z_0)^{0.67}. \quad (12)$$

Beljaars (1987, personal communication) found using a  $k$ - $\epsilon$  model a power law with the exponent ( $\propto 0.8$ ) between the above values. Here we adopt Eq. 12 with  $a_4 = 0.3$ .

### 3. NUMERICAL METHODS

The integrals in Eq. 8 must be evaluated numerically. The integration over azimuth can be rewritten as:

$$\int_0^{2\pi} \nabla h(r, \phi) \exp(-i\phi) d\phi = (H_x, H_y) \quad (13)$$

with

$$\begin{aligned} H_x &= \frac{1}{2\pi} \left[ \frac{d}{dr} (I_{cc} - iI_{cs}) + \frac{1}{r} ((I_{cc} - I_{ss}) - 2iI_{cs}) \right] \\ H_y &= \frac{1}{2\pi} \left[ \frac{d}{dr} (I_{sc} - iI_{ss}) + \frac{1}{r} (2I_{cs} - i(I_{ss} - I_{cc})) \right] \end{aligned} \quad (14)$$

<sup>1</sup>suggested by Mason (1988), personal communication

and  $I_{cc}$ ,  $I_{cs}$  and  $I_{ss}$  are integrals over  $\phi$  of terrain height  $h$  multiplied with  $\cos^2(\phi)$ ,  $\cos(\phi)\sin(\phi)$  and  $\sin^2(\phi)$  respectively.

The integration is performed numerically on a polar "zooming" grid with the center point located at a selected point of interest. The polar grid has 72 rays separated by  $5^\circ$  and 100 radial stations on each ray with each radial interval 6% larger than the previous interval. (Fig. ).

The radial scale is fixed by setting the largest radius  $R$  large enough that the entire digitized map of height contours is contained inside the circle with radius  $R$ . The terrain height is numerically evaluated in each gridpoint and the integration over  $\phi$  is performed by summation. This gives the three functions  $I_{cc}$ ,  $I_{cs}$  and  $I_{ss}$  at each radial station. Using Eq. 14 the two (complex) functions  $H_x$  and  $H_y$  are then obtained. Between gridpoints the functions are assumed to vary linearly and they can therefore be decomposed into a sum of finite elements or "chapeau functions":

$$H_x\left(\frac{r}{R}\right) = \sum_{k=1}^{100} C_k\left(\frac{r}{R}\right) H_x\left(\frac{r_k}{R}\right) \quad (15)$$

The  $C_k$  functions are continuous and linear in each of the radial intervals, and  $C_k\left(\frac{r_m}{R}\right) \equiv \delta_{km}$ .

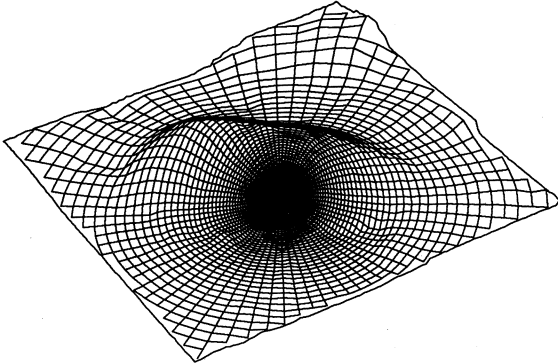


Figure 1: Illustration of the "zooming" polar computational grid.

The radial integration in Eq. 8 is performed analytically for each of the  $C$ -functions. Since they are independent of the terrain heights and the size of the grid these integrals can be precomputed.

Defining the matrix  $M_{jk}$  as

$$M_{jk} = -\frac{2}{c_j J_2^2(c_j)} \int_0^1 \rho C_k(\rho) J_1(c_j \rho) d\rho \quad (16)$$

the evaluation of the expansion coefficients  $\alpha_j$  becomes: matrix multiplication:

$$\alpha_j^x = R \sum_k M_{jk} H_x(r_k), \quad (17)$$

and similarly for  $\alpha_j^y$ .

Because of the zooming grid there appears approximately 2500 nonvanishing terms in the Fourier Bessel expansion (i.e. 250.000 terms in  $\mathbf{M}$ ). Noting however, that the division of the  $\alpha$ 's in the index  $j$  is only done in order to stratify in the length scales  $L$  and  $\ell$ , and noting further that these scales only enter as scale heights into the windprofile which depends only logarithmically (or nearly logarithmically) of height, we can contract the matrix by adding neighbouring elements in the same column  $k$  as long as the length scales do not differ too much. By accepting a relative error in the length scales  $L$  of 5% the number of elements are reduced through this contraction to 6200. The analytical evaluation of the Bessel-functions and integrals in Eq. 16 and the contraction is as mentioned done only once, and the table of the 6200 matrix elements is simply an data block which is input to the model. This procedure reduces the required computing time from hours to 3-4 seconds (on a 20MHz PC).

The input to the model is a file of digitized terrain contours, that is strings of coordinates  $(x, y)$  on each line of constant terrain height. Using a fast sorting routine all the gridpoint terrain heights are evaluated with only one pass of the terrain contour file.

#### 4. MODEL RESULTS.

The model has been integrated into a general wind energy resource assessment model (the Wind Atlas Analysis and Application Model WA<sup>3</sup>P, see Troen and Petersen, 1989)) in which also inhomogeneities in surface roughness are treated, and this model has been used extensively for the calculation of mean flow perturbations in complex terrain.

Here we will illustrate the model performance by looking at the model prediction and the measured data for the mean flow over the Askervein Hill. The hill is illustrated by the computer generated "photo" in fig. , and the details of the field experiment from which the measured data where obtained is reported in Taylor and Teunissen (1987), and Salmon et. al. (1987). In fig. measured and model predicted values of the relative change of the mean speed at 10m above ground level for a upstream wind direction of  $210^\circ$  is shown. One notes the excellent agreement between model and measurements on the upwind side and on the hilltop, and that this model as other linearized models have difficulties on the downwind side of this (and other) steep hill(s).

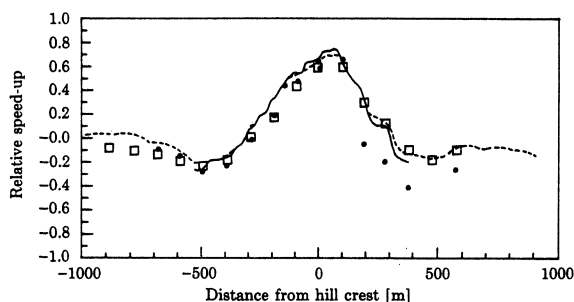


Figure 2: Relative speed-up ratios for the flow along the A-line across the Askervein Hill. Measurements are indicated by filled dots and results from the present model by open squares. Results from two other models are shown by the full and dashed lines (from Beljaars et. al. 1986)

Further development of the present model is concentrating on removing this deficiency. Other results from application of the model and discussion of the accuracy of the model is presented in Jensen et al (1990, these proceedings) and in Walmsley et al (1990).

## 5. CONCLUSION

A high resolution three dimensional spectral model for mean flow calculation in complex terrain is described. Compared to other models based on similar physical arguments the present model has several advantages:

- The computational grid concentrates the resolution at a single point, making it possible to obtain effective values of horizontal resolution of the order of a few meters or less and still adequately model features out to distances of tens of kilometers.
- The localization at one single point makes it possible to include the effects of large inhomogeneities in the surface roughness taking into account both the effects on the upwind profile and on the inner length scale ( $\ell$ ).
- The combination of an extremely effective spectral solution technique and a fast sorting algorithm makes it possible to calculate a windprofile over complex terrain from grid independent digitized height contours in a matter of seconds on a desktop PC.

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